Learning the Graph of Epidemic Cascades

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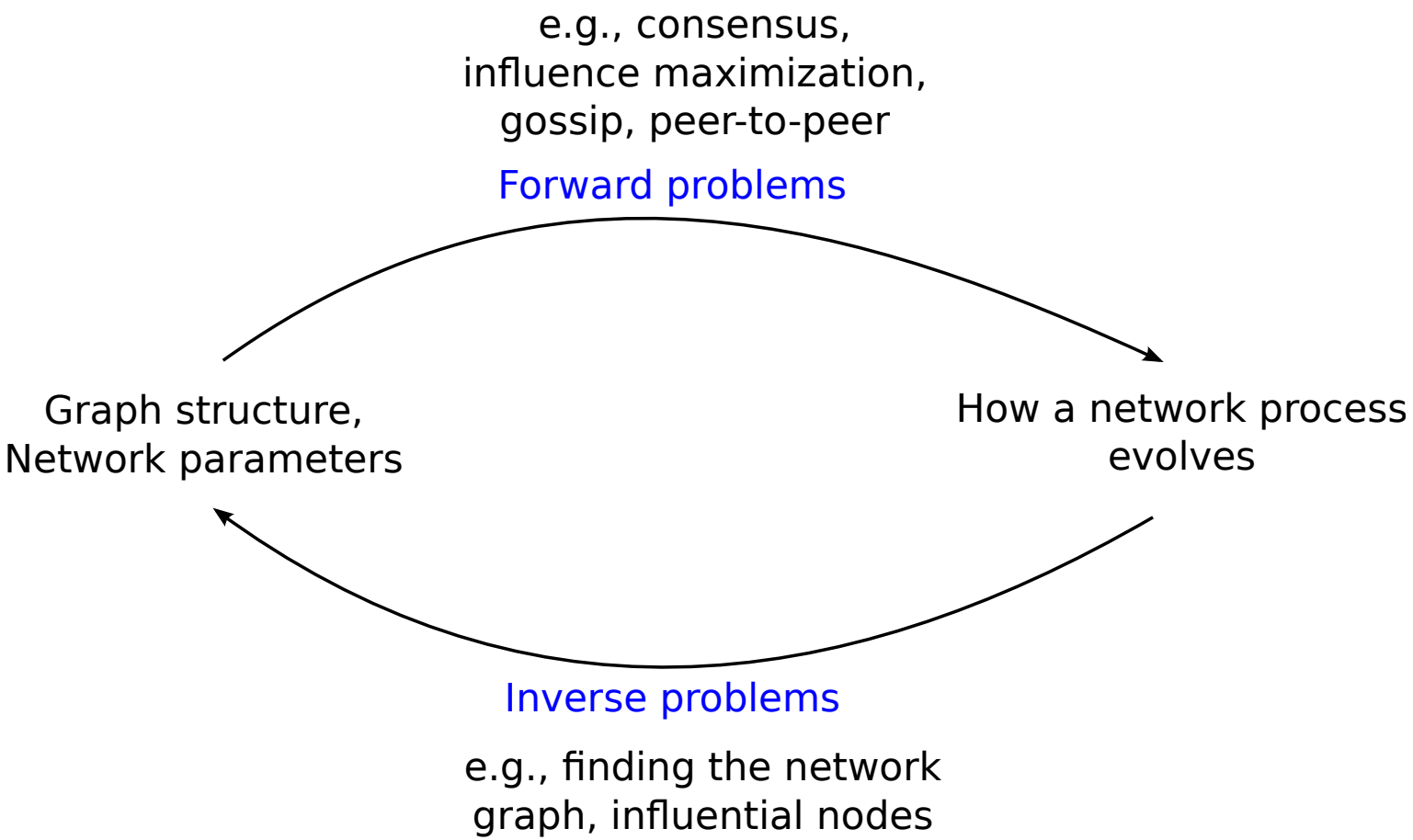
Two types of problems



Today: finding the network graph of cascades

Cascades

- Cascades are processes where nodes get infected and further infect other nodes and so on.
- They are used to model the spread of
  - infectious diseases
  - online media such as videos, news stories etc
  - offer uptakes on social buying sites such as Groupon, Living Social etc.

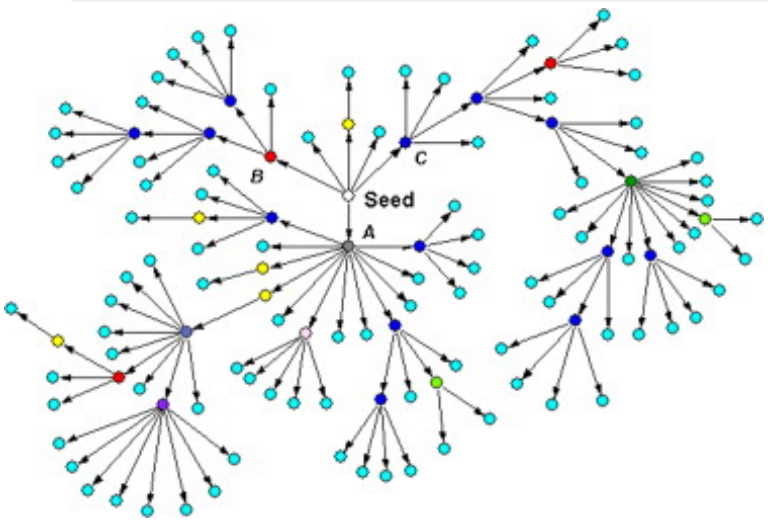
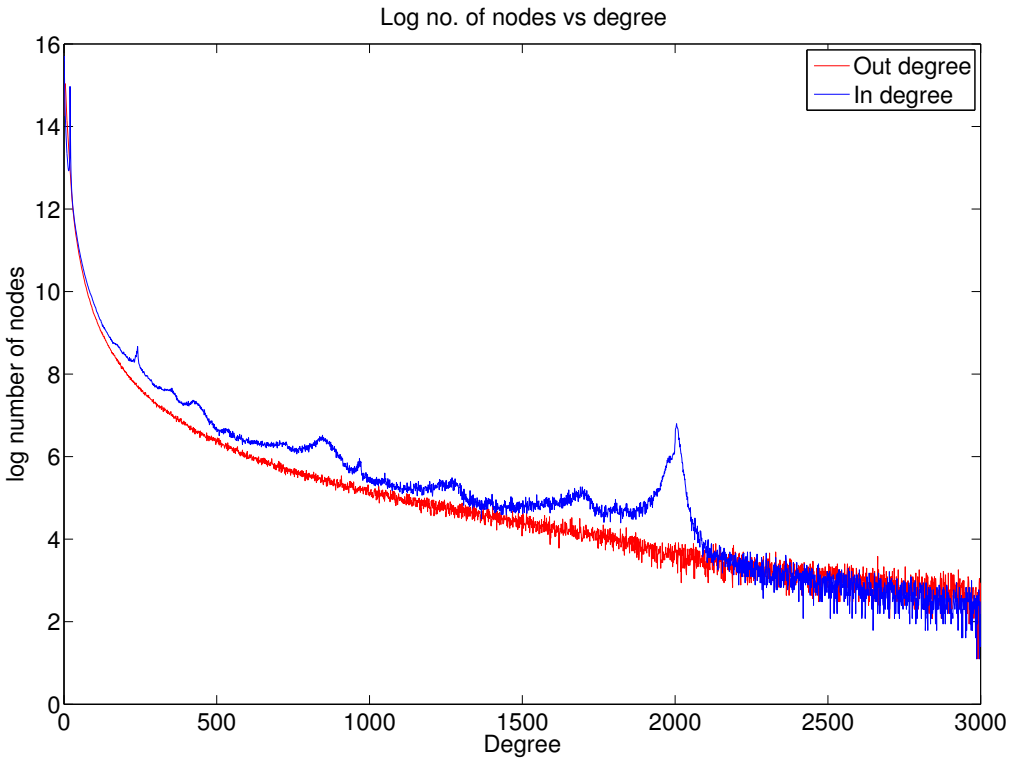


Fig : A tree cascade
Courtesy : Sciencedirect

Motivation for Graph Learning

- Require knowledge of the graph for solving forward problems
- Presence of rubbish edges in online social networks

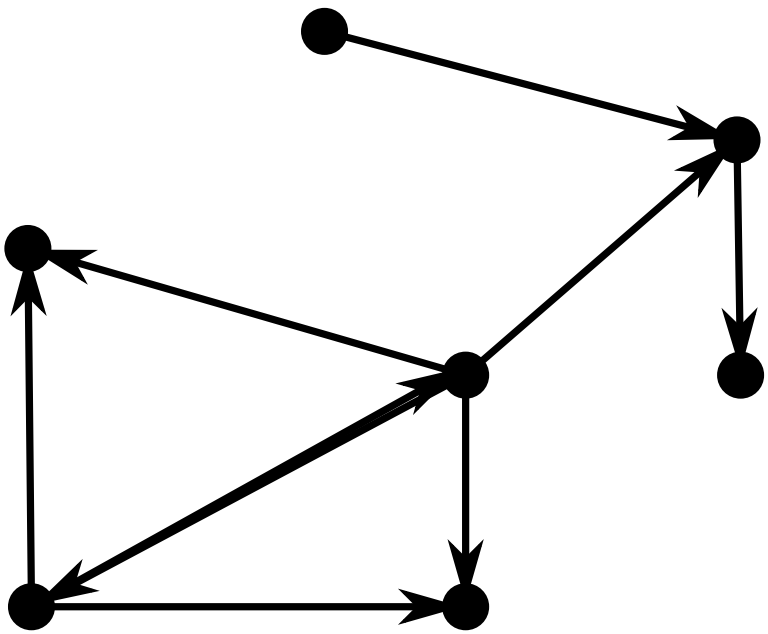


Twitter degree plot - more than 10 mn users have degree > 100

Overview

1. Standard Independent Cascade Model
2. Graph Learning Problem
3. Algorithms
4. Lower Bounds

Standard Independent Cascade Model[1]



- discrete time
- initial seeds random
- i infects j w.p. p_{ij}
- inactive after one time step

[1] Kempe, Kleinberg and Tardos, 2003.

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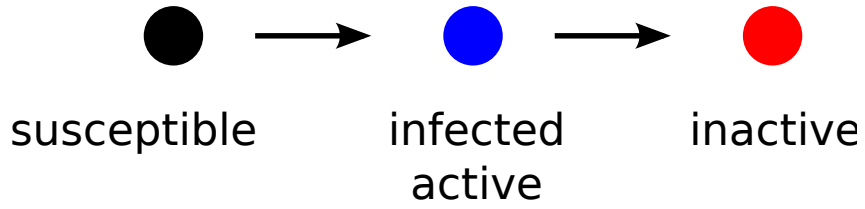

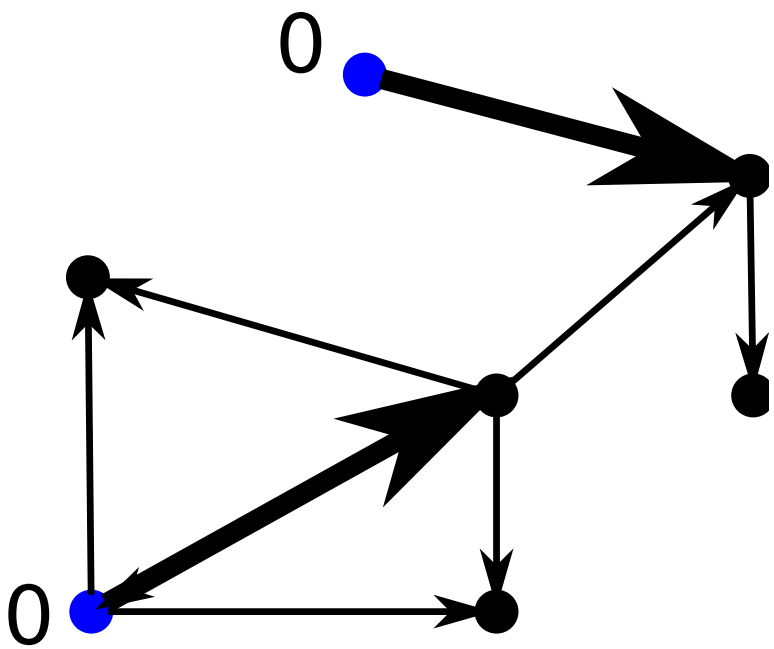

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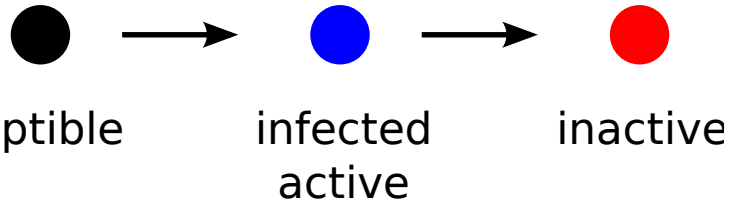

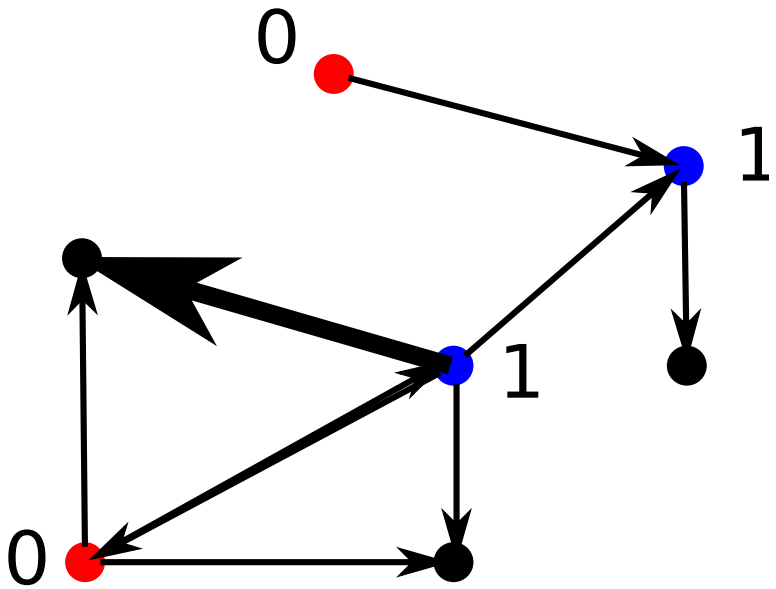

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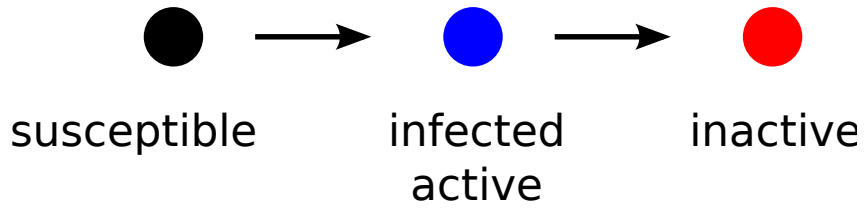

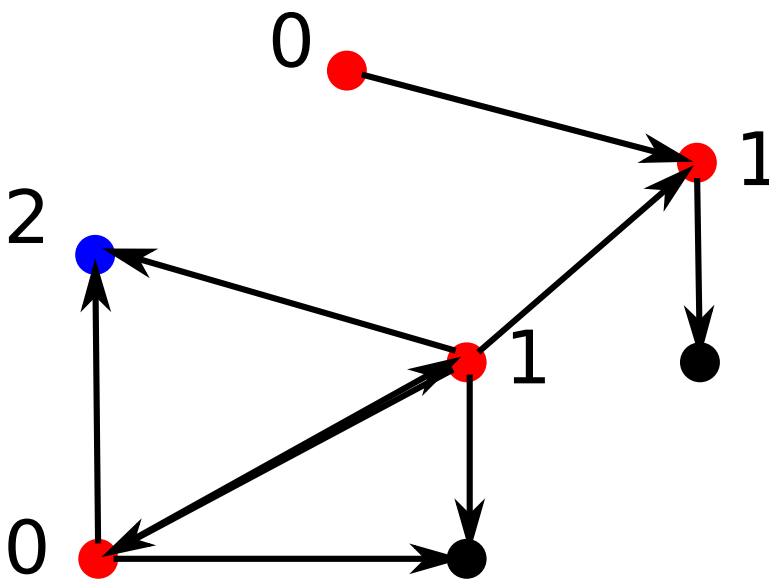

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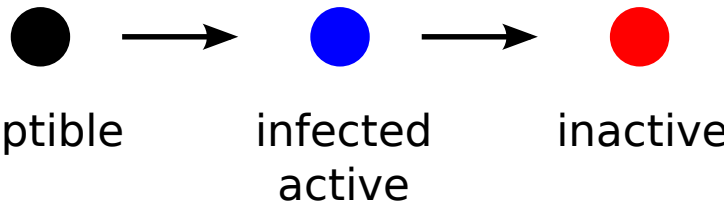

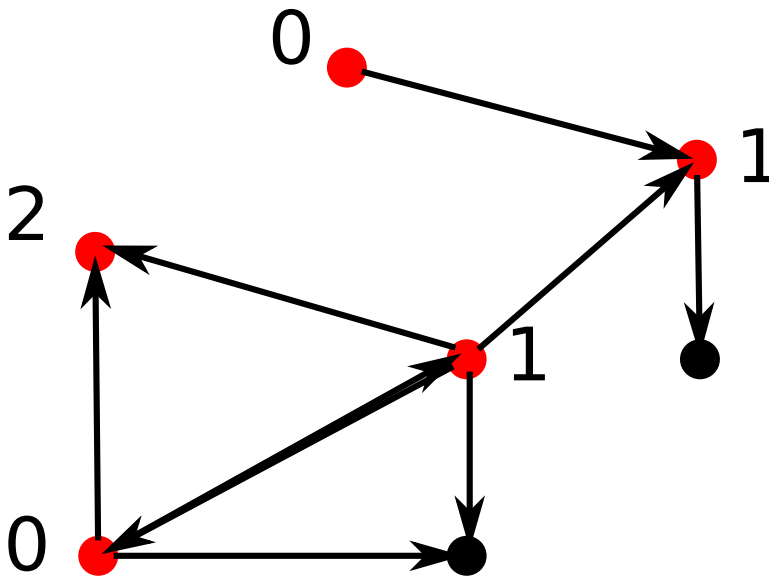

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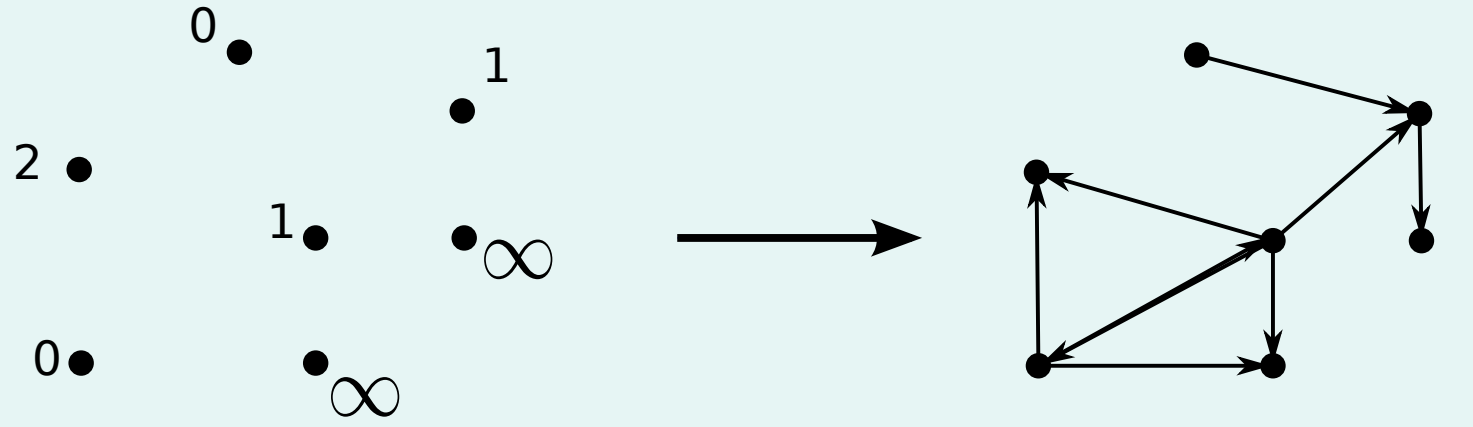
Graph learning problem



Given node activation times

Find network graph

- clearly not possible with a single cascade
- super graph may be available

Overview of Results

- Computational (fast algorithms)
  - show that Maximum likelihood becomes a decoupled convex program ⇒fast distributed implementation
- Statistical (how many cascades do we need to obtain the truth?)
  - performance of Maximum likelihood algo.
  - performance of Greedy algo.
  - Information theoretic lower bounds

Maximum likelihood

$$\mathcal{L}(t,p) = \log \mathbb{P}\left[t|p\right]$$

infection times

edge probabilities

Classical Results

- $\widehat{p}(t) := \arg\max_p \mathcal{L}(t,p)$
- Consistency : As $m \to \infty$ we have $\widehat{p}(T) \to p^*$

However, in our setting

- Computational: efficient maximization of likelihood?
- Statistical: correctness when $m << n$ (high dimensional scaling)?

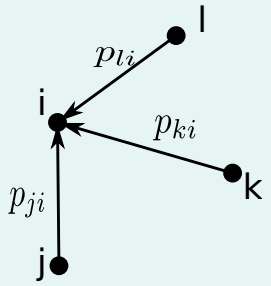
Maximum likelihood

Our insights into the log likelihood function

- **Decouples** into *independent* node wise problems

$$\mathcal{L}(t,p) = \sum_i \mathcal{L}_i(t, p_{*i}) \text{ where}$$

$$\mathcal{L}_i(t, p_{*i}) = \log \mathbb{P}\left[t_i \middle| p_{*i}, t_{\setminus i}\right]$$



- Becomes **concave** after change of variables

$$\theta_{ij} = -\log(1 - p_{ij})$$

Finding the graph $\Rightarrow$ Solving a convex problem for each node (in parallel)

Maximum likelihood

ML based algorithm for node i

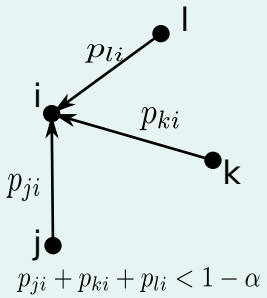
- Maximize log likelihood: $\widehat{\theta}_{*i} := \arg\max_{\theta_{*i}} \mathcal{L}_i(t, \theta_{*i})$
- Threshold: $\widehat{\mathcal{N}}_i = \left\{ j \middle| \widehat{\theta}_{ji} > \eta \right\}$

So : resolved computational issues
Now : focus on statistical issues - how many cascades do we need to learn the graph?

Setting

- Each node has probability p_{init} of being a seed
- Each edge is strong enough
- Correlation decay $\sim$ far away nodes have negligible effect

Require $\sum_j p_{ji} < 1 - \alpha \; \forall \; i$
Distance between seeds and infected nodes not huge



Consequence :
$p_{\text{init}} <$ Prob of participating in a cascade $< \frac{p_{\text{init}}}{\alpha}$

Main Result

Theorem

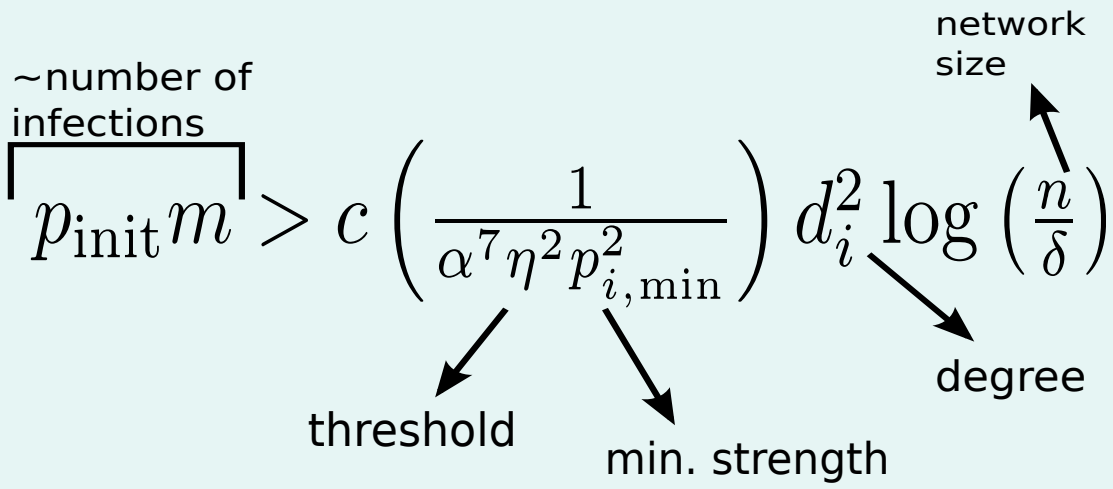
For node i with parental degree d_i, if the total number of cascades is

$$m > \frac{c}{p_{\text{init}}} \left(\frac{1}{\alpha^7 \eta^2 p_{i,\min}^2} \right) d_i^2 \log \left(\frac{n}{\delta} \right)$$

then w.p. at least $1 - \delta$, the ML algorithm gives

1. no false parents and
2. all strong enough parents (i.e., $p_{ji} > \frac{8}{\alpha} \left(e^{2\eta} - 1 \right)$)

Comments



- Non asymptotic
- Node by node guarantees
- entire graph recovery via union bounds
- stronger neighbors easier to identify

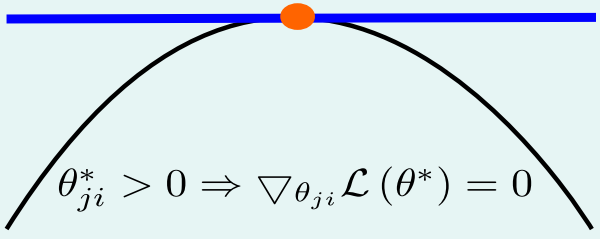
Proof

Main idea

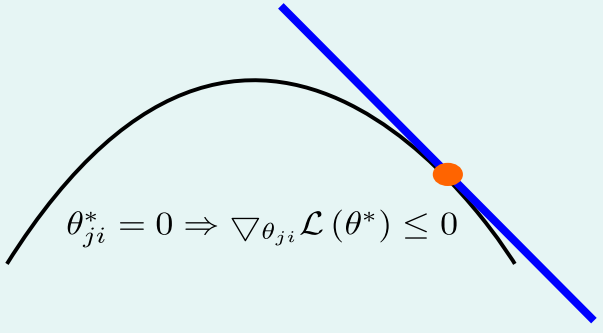
Convex problems $\rightarrow$ first order optimality conditions $\rightarrow$ statistical guarantees

Ideal case

If we had ∞ samples, $\theta^* = \arg\max \mathcal{L}(\theta)$ and so



j is a parent of i



j is NOT a parent of i

Can infer edges from gradient.

Proof

Issue

Finite samples

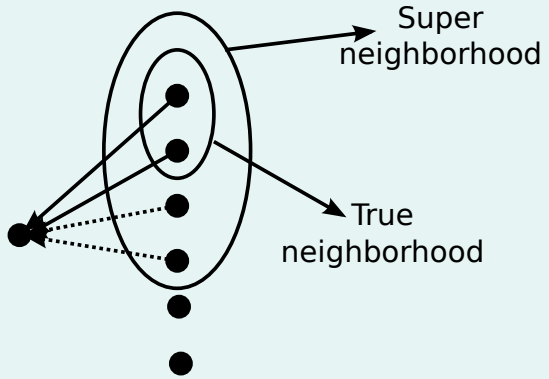
The proof has two steps.

1. **Concentration**: $\bigtriangledown \widehat{L}(\theta^*)$ approximately satisfies the above conditions (via Azuma-Hoeffding)
2. **Smoothness & Convex analysis**: $\theta^* \approx \widehat{\theta}$

Super Graph

Often, we know a super-set of the set of edges.

e.g. Social networks: the entire set of a users online acquaintances
Epidemiology: restrictions based on geography, etc.



This means we have a super set of parents for each node.

ML Algorithm

same as before but
optimize only within super neighborhood
needs only local info

Super Graph

Theorem

For recovery of node i's parents, need

$$m > \frac{c}{p_{\text{init}}} \left(\frac{1}{\alpha^7 \eta^2 p_{i,\min}^2} \right) d_i^2 \log \left(\frac{D_i}{\delta} \right)$$

Comments

- D_i : size of the super-neighborhood
- Sample complexity independent of n
- Computational complexity reduces as well!

Greedy Algorithm

Recall : j can infect i in only one time step after j is infected.

For node i, start with all cascade samples

Iteratively

Pick the node which gets active one time-step before i in the most number of remaining samples

Remove that node and corresponding cascades

Repeat until all cascade samples exhausted

Nodes	i	j	k
Cascade 1	0	1	1
Cascade 2	1	1	0
Cascade 3	2	1	0
Cascade 4	1	0	1
Total	-	2	1

Nodes	i	k
Cascade 1	0	1
Cascade 2	1	0
Total	-	1

Greedy Algorithm

Theorem

If the graph is a tree, just need

$$m > \frac{c}{p_{\text{init}}} \left(\frac{1}{p_{i,\min}} \right) d_i \log \left(\frac{D_i}{\delta} \right)$$

- linear in degree
- correlation decay not needed

Information theoretic lower bounds

Theorem

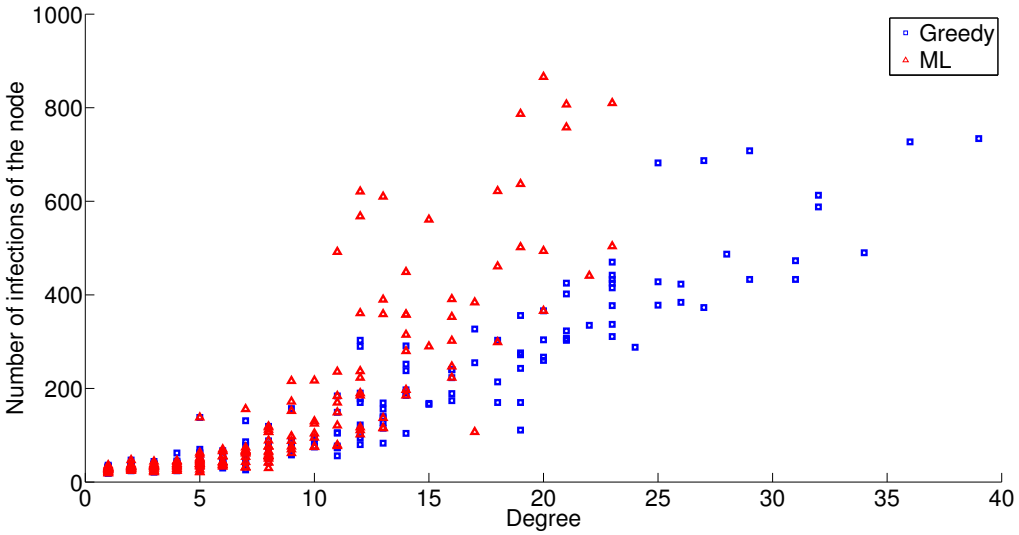
Set of graphs == all with in-degree at most d.
Want: recovery of the graph with prob. at least $1 - \delta$. Then need:

$$m \gtrsim d \log\left(\frac{D}{d}\right)$$

Comments

- Shows ML tight upto a d factor.
- Extended results upto some edit distance γ

Dependence on degree



- twitter sub-graph
- dependence super-linear for ML
- seems linear for greedy though graph is not a tree

Summary

Graph learning possible

- from small no. of cascades
- via fast distributed algorithms
- using local info

Conclusions

Inverse problems: Infer network structure, properties by observing network processes.

- increasingly important
- require new tools and methods

Thanks!