Provable Matrix Completion using Alternating Minimization

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Alternating Minimization (AltMin)

**General Algorithm**

To minimize $f(X)$ over rank-$k$ matrices $X$, repeat the following:
- fix $U$ and minimize $f(UV^\dagger)$ over $V$
- fix $V$ and minimize $f(UV^\dagger)$ over $U$

A popular Empirical approach to solve low rank matrix problems eg. matrix completion, clustering etc.

**Challenge**: few theoretical guarantees
Matrix Completion

- Given some elements, fill in the rest
- Not possible in general; what if low rank?
- Metrics: Sample complexity and Computational complexity
Matrix Completion via Alternating Minimization

\[
\begin{align*}
\min & \sum_{(i,j) \in \text{known set}} (M_{ij} - X_{ij})^2 \quad \text{s.t.} \quad \text{rank}(X) \leq k \\
= & \min \sum_{(i,j) \in \text{known set}} \left( M_{ij} - U_i^\dagger V_j \right)^2 \quad \text{s.t.} \quad U \in \mathbb{R}^{m \times k}, \ V \in \mathbb{R}^{n \times k}
\end{align*}
\]
A Comparison

- **Nuclear norm / Trace norm approach**: convex relaxation.
- **Empirically**, AltMin has
  - similar sample complexity and
  - better computational complexity.

**Challenge**: AltMin formulation is non-convex.
Our Results

- First theoretical guarantees for AltMin in any low rank setting
- We prove results for
  - matrix sensing
  - matrix completion
Problem: Given $y$ and $A$, recover $X$.

Natural Algorithm (AltMinSense)

1. (Initialization) $\hat{U}^0 \leftarrow$ top $k$-left s.v. of $\sum y_i A_i$
2. In iteration $t$:
   - $\hat{V}^t \leftarrow \arg\min_{V \in \mathbb{R}^{n \times k}} \| y - A(\hat{U}^{t-1} V^\dagger) \|_2$
   - $\hat{U}^t \leftarrow \arg\min_{U \in \mathbb{R}^{m \times k}} \| y - A(U(\hat{V}^t)^\dagger) \|_2$
Restricted Isometry Property (RIP)

Existing results require RIP assumptions.

**RIP [RFP10]**

A linear operator $\mathcal{A}() : \mathbb{R}^{m \times n} \to \mathbb{R}^d$ satisfies $k$-RIP with $\delta_k$, if for all $X \in \mathbb{R}^{m \times n}$ s.t. rank$(X) \leq k$, the following holds:

$$(1 - \delta_k) \|X\|_F^2 \leq \|\mathcal{A}(X)\|_2^2 \leq (1 + \delta_k) \|X\|_F^2.$$ 

- $\delta_k = 0 \Rightarrow$ Identity map
- $\delta_k = 1 \Rightarrow$ No information
Existing Results

**Trace norm approach [RFP10]**

\[
\begin{align*}
\min_{X} \|y - \mathcal{A}(X)\|_2 \quad &\rightarrow \quad \min_{X} \|y - \mathcal{A}(X)\|_2 \\
\text{s.t.} \quad \text{rank}(X) \leq k \quad &\rightarrow \quad \text{s.t.} \quad \|X\|_* \leq \sqrt{k}
\end{align*}
\]

- \(\delta_{5k} < \frac{1}{10}\)

**Singular Value Projection [JMD10]**

- \(\delta_{2k} < \frac{1}{3}\)

**Drawback**

- Need to compute many SVDs during execution - very slow in practice
Our Results

Theorem

If $\delta_{2k} < \left( \frac{\sigma_k}{\sigma_1} \right)^2 \frac{1}{100k}$, then

$$\| M - \hat{U}^T (\hat{V}^T) \|^F < \left( \frac{1}{2} \right)^T$$

Remarks

1. $\delta_{2k}$ depends on the condition number unlike in existing work
   - modified algorithm: $\delta_{2k} < \frac{1}{3200k^2}$
2. Linear convergence: $\log \frac{1}{\epsilon}$ iterations for $\epsilon$ error.
Matrix Completion

Problem
Given elements in $\Omega$, find the low rank matrix $M$.

Analysis is harder
- $\Omega$ does not in general satisfy RIP.
- Dependence between iterates.
Our Algorithm

- Divide $\Omega$ into $2T + 1$ subsets $\Omega_0, \ldots, \Omega_{2T}$ by uniform sampling.
- Use $\Omega_i$ for the $i^{th}$ iteration of AltMin.

AltMinComplete

(Initialization) $\hat{U}^0 \leftarrow$ top $k$-left s.v. of $(M)_{\Omega_0}$

FOR $t = 0, \ldots, T - 1$

\[
\hat{V}^{t+1} \leftarrow \arg\min_{V \in \mathbb{R}^{n \times k}} \left\| \left( \hat{U}^t V^\dagger - M \right)_{\Omega_{t+1}} \right\|_F^2
\]

\[
\hat{U}^{t+1} \leftarrow \arg\min_{U \in \mathbb{R}^{m \times k}} \left\| \left( U \left( \hat{V}^{t+1} \right)^\dagger - M \right)_{\Omega_{T+t+1}} \right\|_F^2
\]

ENDFOR

- **Conjecture**: Do not need this partition.
- Same algorithm proposed and analyzed independently by [Kes12]
A Hard Case

\[ \begin{array}{cccc}
? & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \]
Incoherence

\[ M = U^* \Sigma^* (V^*)^\dagger \] is incoherent with parameter \( \mu \) if

- \( \| u^{(i)} \|_2 \leq \frac{\mu \sqrt{k}}{\sqrt{m}} \forall i \in [m] \) and
- \( \| v^{(j)} \|_2 \leq \frac{\mu \sqrt{k}}{\sqrt{n}} \forall j \in [n] \).

<table>
<thead>
<tr>
<th>coherent</th>
<th>incoherent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0</td>
<td>0 -1</td>
</tr>
<tr>
<td>0 1</td>
<td>1/\sqrt{n} -1/\sqrt{n}</td>
</tr>
<tr>
<td>0 0</td>
<td>1/\sqrt{n} -1/\sqrt{n}</td>
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<tr>
<td>0 0</td>
<td>1/\sqrt{n} 0</td>
</tr>
<tr>
<td>0 0</td>
<td>1/\sqrt{n} 1/\sqrt{n}</td>
</tr>
</tbody>
</table>
Existing Results

Existing results assume uniform sampling and incoherence of $M$.

Trace norm approach [CR09, CT09]

\[
\min \sum_{(i,j) \in \Omega} (X_{ij} - M_{ij})^2 \quad \text{s.t.} \quad \text{rank}(X) \leq k
\]
\[
\rightarrow \min \sum_{(i,j) \in \Omega} (X_{ij} - M_{ij})^2 \quad \text{s.t.} \quad \|X\|_* \leq \sqrt{k}
\]

- $O(\text{knlog}n)$ observations
- Drawback: need many SVD calculations

OptSpace [KMO10]

- Opt. on Grassman manifold: $O\left(f\left(\frac{\sigma_1^*}{\sigma_k^*}\right) \text{ kn log } n\right)$
- rate of convergence not known
Our Results

Theorem

Let $M$ be incoherent. If

\[ \# \text{ measurements} > C \left( \frac{\sigma_1^*}{\sigma_k^*} \right)^6 k^7 n \log n \log \frac{1}{\epsilon}, \]

then after $T = O \left( \log \frac{1}{\epsilon} \right)$ iterations, we have:

\[ \| M - \hat{U}^T (\hat{V}^T)^\dagger \|_F < \epsilon. \]

Advantages:

- linear convergence : $\log \frac{1}{\epsilon}$ vs $\frac{1}{\sqrt{\epsilon}}$
- each iteration very fast
- low storage requirement

Weakness: Dependence on

- condition number
- required accuracy
- $k$
## Comparison

<table>
<thead>
<tr>
<th></th>
<th>Sample comp. ((d))</th>
<th>Comp. comp.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Our Results</strong></td>
<td>(O \left( \left( \frac{\sigma_1^<em>}{\sigma_k^</em>} \right)^6 k^7 n \log n \log \frac{1}{\epsilon} \right))</td>
<td>(O \left( dk^2 \log \frac{1}{\epsilon} \right))</td>
</tr>
<tr>
<td>AltMin [Kes12]</td>
<td>(O \left( \left( \frac{\sigma_1^<em>}{\sigma_k^</em>} \right)^8 kn \log n \log \frac{1}{\epsilon} \right))</td>
<td>(O \left( dk^2 \log \frac{1}{\epsilon} \right))</td>
</tr>
<tr>
<td>Trace norm [CT09]</td>
<td>(O \left( kn \log n \right))</td>
<td>(O \left( \frac{n^3}{\sqrt{\epsilon}} \right))</td>
</tr>
</tbody>
</table>

AltMin: Provable Matrix Completion using Alternating Minimization

Trace norm: Provable Matrix Completion using Trace Norm Minimization
Main Idea of the Proof

- If $\Omega = \text{all elements}$, then AltMin becomes the well-known power method.
- In general, iterates take the form:

  \[ \hat{V}^{t+1} = V^* \Sigma^* U^\dagger U^t - F \]  

  and

  \[ \|F\|_2 \downarrow \text{ as } t \uparrow. \]

- Use RIP to show decay.
- Technical difficulty: Establishing incoherence of $U^t$.  


Summary

- First theoretical guarantees for AltMin in any low rank setting
- Results for
  - Matrix sensing
  - Matrix completion

Further Directions

- Recent result for AltMin in Phase retrieval [NJS13]
- Theory for AltMin in clustering, sparse PCA, NMF etc.?
References

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