One-bit Compressed Sensing: Provable Support and Vector Recovery

Praneeth Netrapalli

The University of Texas at Austin

Joint work with **Sivakant Gopi**, **Prateek Jain** and **Aditya Nori**

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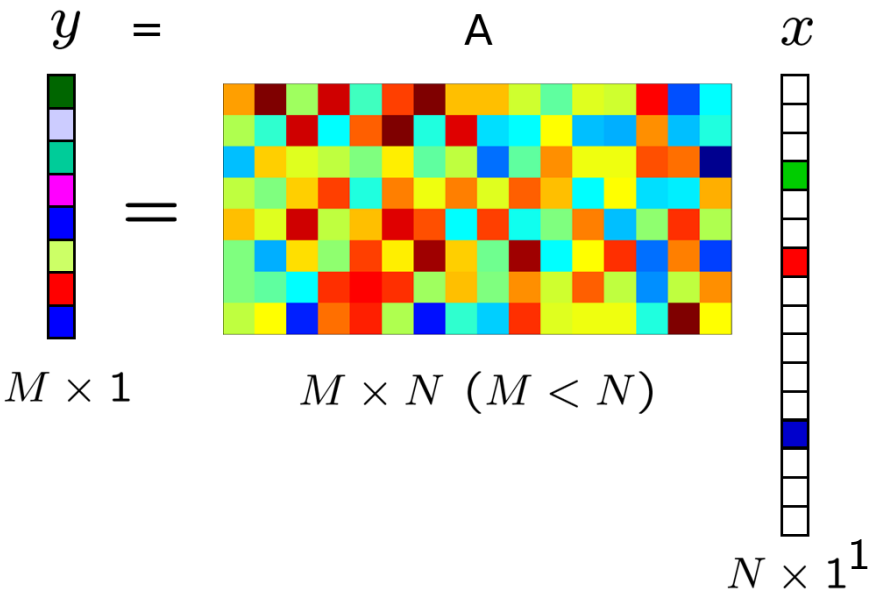
Compressed Sensing



[1]

- Goal: Reconstruct a sparse signal using very few linear measurements
- Tremendous amount of work in the last decade
- $O(k \log n)$ measurements to reconstruct k-sparse signals in $\mathbb{R}^n$

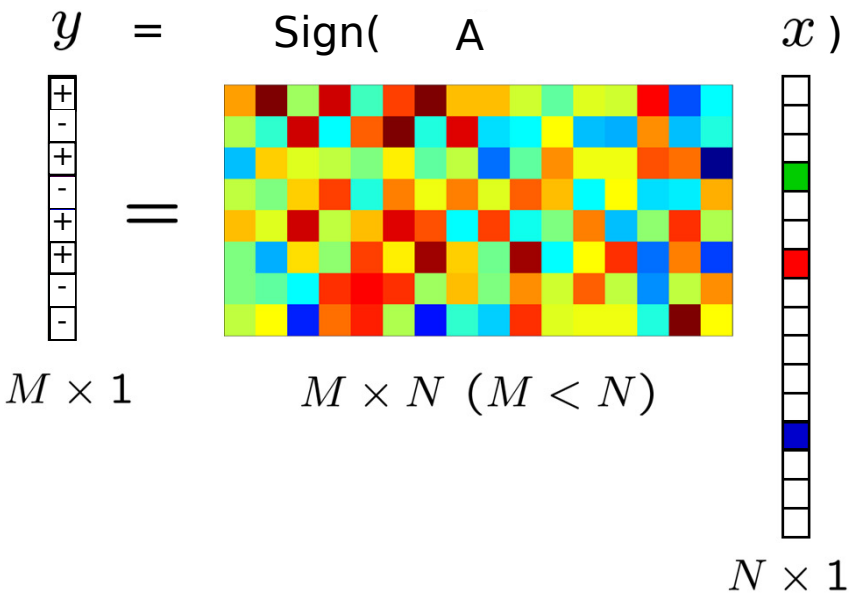
[1] http://lions.epfl.ch/research

Quantization

- Measurements up to infinite precision - not practical
  - $y_j = -1.010001011110$
- Arbitrary quantization does not work well [BB08]
  - $y_j = -1.01$
- Extreme quantization - single bit measurements
  - $y_j = -1$

One-bit Compressed Sensing [BB08]

Goal: Reconstruct a sparse signal using *signs* of very few linear measurements



- Formally, given y and A, recover x
- Motivation:
  - Captures extreme quantization
  - Easy to implement
  - Robust to noise

Frameworks

Exact recovery not possible:

$$\text{For example, } y = \text{Sign}(Ax) = \text{Sign}(A(cx)),\ \forall c > 0$$

Frameworks

- Support recovery: recover $\widehat{x}$ such that

$$\text{Supp}(x) = \text{Supp}(\widehat{x})$$

- ϵ-approximate recovery[a]: recover k-sparse $\widehat{x}$ such that

$$\left\| \widehat{x} - \frac{x}{\|x\|} \right\| < \epsilon$$

[a] $\|\cdot\|$ refers to two norm

Metrics

- Measurement (or sample) complexity: dimension of y
- Computational complexity: time taken by the recovery algorithm
- Universality: use the same measurement matrix A for all sparse vectors x
  - critical for many applications e.g., single pixel camera

Our Results – Support Recovery

Algorithm	[HB11]	UFF	Expanders

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# measurements	$O(k \log n)$	$O\left(k^2 \log n\right)$	$O\left(k^3 \log n\right)$
Running Time	$O(n \log n)$	$O(nk \log n)$	$O(nk \log n)$

- *First universal* measurement schemes
- Open problem: $O(k \log n)$ universal measurement scheme not known

Union Free Family

Definition (k-Union Free Family)

Sets $\mathcal{F} := \{B_1, \cdots, B_n\}$*:*

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$$B_{i_0} \nsubseteq B_{i_1} \cup \cdots \cup B_{i_k},$$

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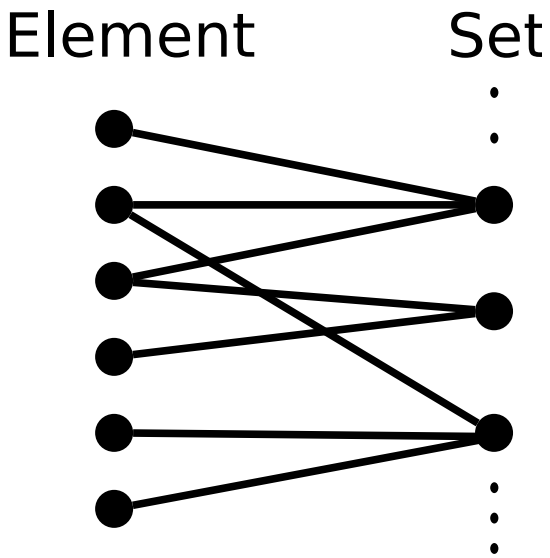

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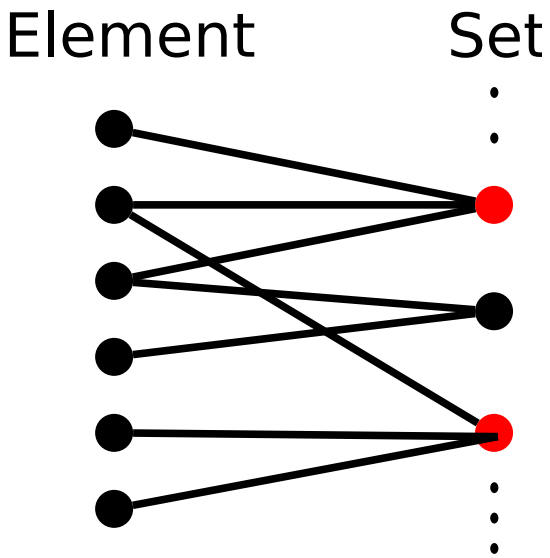

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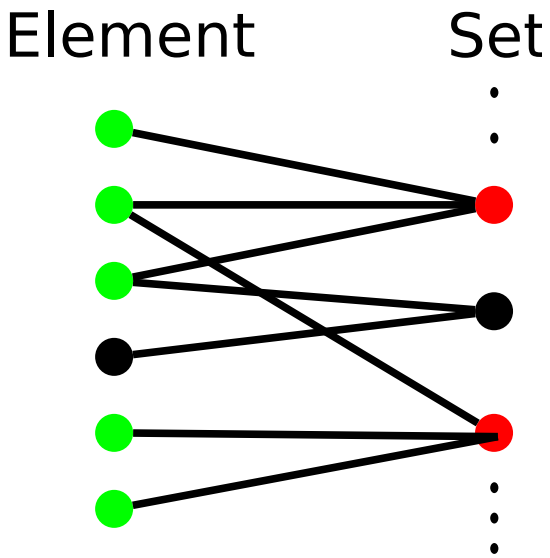

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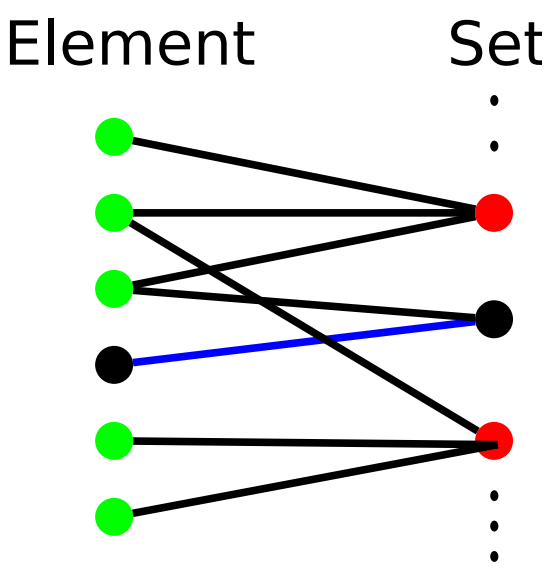
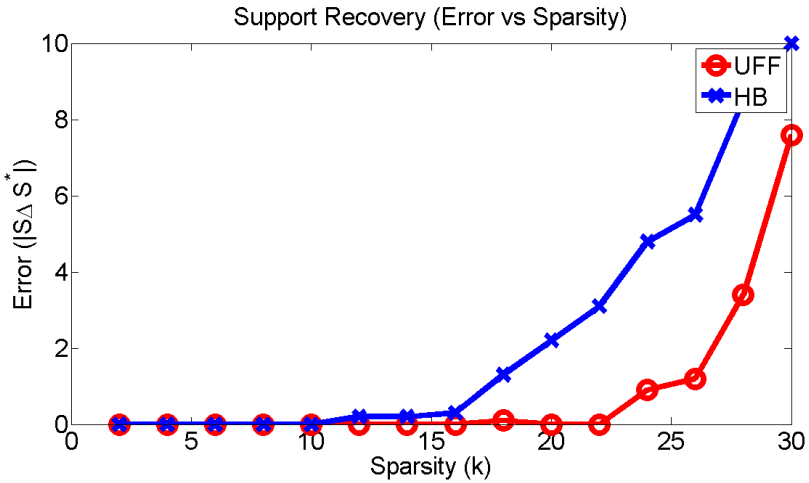
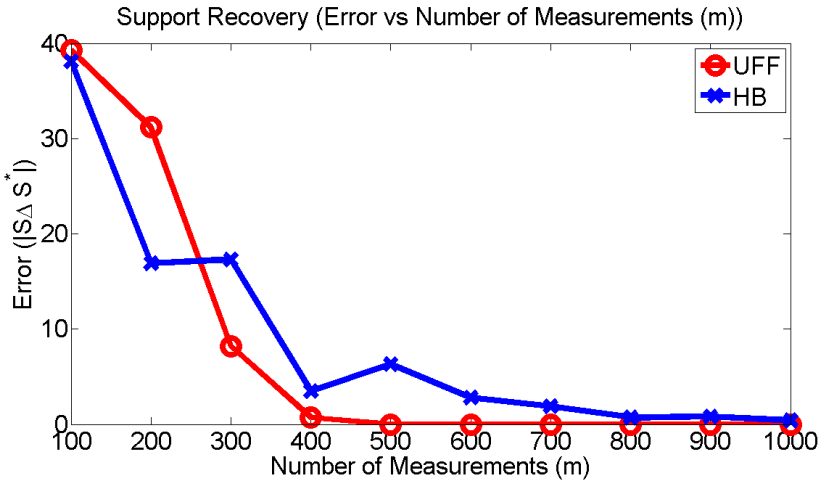


- Well studied combinatorial objects
- Use their structure in the measurement scheme *and* recovery algorithm
- Works for $x \geq 0$
  - Another algorithm based on expanders, which works for any x

Experiments





ϵ-approximate recovery

- Recall: Need to recover $\widehat{x}$ such that $\left\| \widehat{x} - \frac{x}{\|x\|} \right\| < \epsilon$
- Plan and Vershynin [PV11, PV12]: $O\left(\epsilon^{-5}\right)$ measurements
  - hard thresholding and soft thresholding based approaches
- Our result: $\widetilde{O}\left(\epsilon^{-1}\right)$ measurements
- Key idea: Combine optimal results from compressed sensing and learning halfspaces

Our Results – Approximate Recovery

Algorithm	[PV11, PV12]	Two-stage	S-Approx

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# measurements (m)	$\frac{k}{\epsilon^5} \log^2 \frac{n}{k}$	$\frac{k}{\epsilon} \log \frac{n}{k}$	$k^3 \log \frac{n}{k} + \frac{k}{\epsilon}$

Our Results – Approximate Recovery

Algorithm	[PV11, PV12]	Two-stage	S-Approx
Universal	Yes	Yes	Yes
# measurements (m)	$\frac{k}{\epsilon^5}\log^2\frac{n}{k}$	$\frac{k}{\epsilon}\log\frac{n}{k}$	$k^3\log\frac{n}{k}+\frac{k}{\epsilon}$
Running Time	$\frac{kn}{\epsilon^6}\log\frac{n}{k}$	$\frac{kn}{\epsilon^5}\log\frac{n}{k}$	$\frac{k^5 n}{\epsilon^5}\log\frac{n}{k}$

- Two-stage: *Near optimal* measurement complexity

Measurement Scheme and Algorithm

$$y = Sign(\overbrace{A_2 A_1}^{A} x)$$

A_2: Random Gaussian Matrix

A_1: Compressed Sensing Matrix

Measurement Scheme and Algorithm

$$y = Sign(\overbrace{A_2 A_1}^{A} x)$$

Random Gaussian Matrix

Compressed Sensing Matrix

Algorithm

1. Linear programming: Obtain $\widehat{z}$ such that $y = \mathrm{Sign}\,(A_2 \widehat{z})$

Measurement Scheme and Algorithm

$$y = Sign(\overbrace{A_2 A_1}^{A} x)$$

Random Gaussian Matrix

Compressed Sensing Matrix

Algorithm

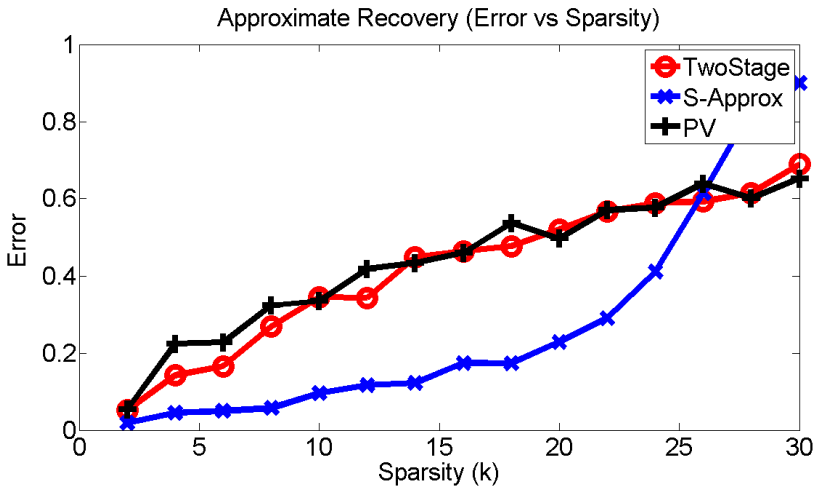
1. Linear programming: Obtain $\widehat{z}$ such that $y = \mathrm{Sign}(A_2 \widehat{z})$
2. Solve CS problem: $\widehat{z} = A_1 x + e$ (GraDeS [GK09])

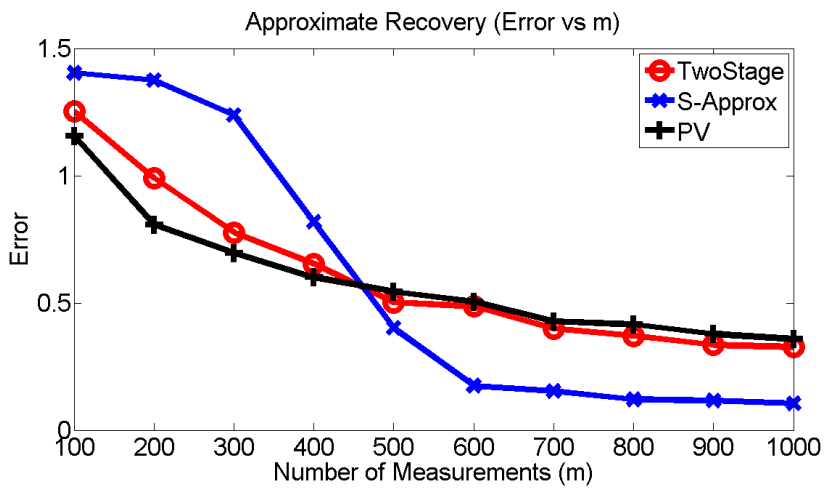
Proof Outline

$$y = Sign(A_2 z)$$
$$\|$$
$$A_1 x$$

1. Robust CS: sufficient to obtain $\widehat{z}$ such that $\|\widehat{z} - z\| < C\epsilon$.
   - requires $O(k \log n)$ measurements
2. Obtain $\widehat{z}$: requires $O\left(\frac{k \log n}{\epsilon} \log\left(\frac{k \log n}{\epsilon}\right)\right)$ random Gaussian measurements.

Experiments





Summary

- One-bit Compressed Sensing
  - captures extreme quantization
  - several other motivations
- Support recovery
  - *First universal* measurement schemes using UFF and expanders
  - UFF: $O\left(k^2 \log n\right)$ measurements but for $x \geq 0$
    - Can be used in other settings?
  - Expanders: $O\left(k^3 \log n\right)$ measurements for any x
  - Open question: achieving $O\left(k \log n\right)$ sample complexity
- ϵ-approximate recovery
  - *Near optimal* sample complexity with ϵ
  - Support recovery based algorithm - works well empirically

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